2.8a stability in first-order systems

Monday, February 8, 2021 12:36 PM

Recall: $x_{t+1} = f(x_t)$ is locally asymptotically stable at an equilibrium $\bar{x}$ if $|f'(\bar{x})| < 1$

unstable at an equilibrium $\bar{x}$ if $|f'(\bar{x})| > 1$.

What is the equivalent for first-order systems?

Thm: Let $X(t+1) = F(X(t))$ be a system of n first-order equations,

$X(t) = (x_1(t), \dots, x_n(t))^T$, $F = (f_1, \dots, f_n)^T$, and $f_i = f_i(x_1, \dots, x_n)$.

Let $\bar{X}$ be an equilibrium of the system. Then linearization of the system about $\bar{X}$ and letting $U(t) = X(t) - \bar{X}$ gives a system

$$U(t+1) = J\,U(t),$$

where J is the Jacobian matrix of F at $\bar{X}$,

$$J(\bar{X}) = \begin{pmatrix} \frac{\partial f_1}{\partial x_1}(\bar{X}) & \frac{\partial f_1}{\partial x_2}(\bar{X}) & \cdots & \frac{\partial f_1}{\partial x_n}(\bar{X}) \\ \vdots & & \ddots & \vdots \\ \frac{\partial f_n}{\partial x_1}(\bar{X}) & \cdots & \cdots & \frac{\partial f_n}{\partial x_n}(\bar{X}) \end{pmatrix}$$

Then $\bar{X}$ is locally asymptotically stable if $|\lambda_i| < 1$ $\forall$ eigenvalues λ_i

and unstable if some $|\lambda_i| > 1$.

Proof sketch: For $X(0)$ s.t. $|\bar{X} - X(0)| < \varepsilon$, with ε sufficiently small,

can approximate $X(t+1) = F(X(t))$ by the Taylor Series of F

$$X(t+1) \approx F(\bar{X}) + \underbrace{J(\bar{X})}_{\text{Jacobian}}(X(t) - \bar{X}) + \cancel{\tfrac{1}{2}(X(t)-\bar{X})^T \underbrace{H(\bar{X})}_{\text{Hessian}}(X(t)-\bar{X})} + \cdots$$

$$\Rightarrow X(t+1) - \bar{X} \approx J(\bar{X})(X(t) - \bar{X}) \quad \text{for } X(t) \text{ sufficiently close to } \bar{X}.$$

$$U(t+1) \approx J(\bar{X})\,U(t)$$

$$\Rightarrow U(t) = [J(\bar{X})]^t\, U(0).$$

If all eigenvalues $|\lambda_i| < 1$, then $\rho(J(\bar{X})) < 1$.

If $\rho(J(\bar{X})) < 1$, then $\lim_{t\to\infty} [J(\bar{X})]^t \to 0$, so $\lim_{t\to\infty} X(t) = \bar{X}$.

If $|\lambda_i| > 1$ for some i, then so long as $[X(0) - \bar{X}] \cdot V_i \neq 0$, where V_i is the eigenvector associated with $|\lambda_i| > 1$, then $\lim_{t\to\infty} |U(t)| = \infty$.

Of course, the linearization may break down as $|X(t) - \bar{X}|$ grows, but $X(t)$ will still leave a sufficiently small ball around $\bar{X}$, so it is unstable. $\square$

Thm 2.10 Let $J \in \mathbb{R}^{2\times 2}$. Then $|\lambda_i| < 1$ $\forall$ eigenvalues λ_i iff

$$|Tr(J)| < 1 + \det(J) < 2.$$

And $|\lambda_i| > 1$ for some eigenvalue λ_i if at least one of the following is true:

$$Tr(J) > 1 + \det(J), \quad Tr(J) < -1 - \det(J), \quad \det(J) > 1.$$

Thm 2.11 (Jury conditions, Schur-Cohn criterion, $n=3$)

Suppose $p(\lambda) = \lambda^3 + a_1\lambda^2 + a_2\lambda + a_3$, $a_1, a_2, a_3 \in \mathbb{R}$ is a polynomial

Then the solutions $\lambda_1, \lambda_2, \lambda_3$ of $p(\lambda) = 0$ satisfy $|\lambda_i| < 1$ iff

(1) $p(1) = 1 + a_1 + a_2 + a_3 > 0$

(2) $(-1)^3 p(-1) = 1 - a_1 + a_2 - a_3 > 0$,

(3) $1 - (a_3)^2 > |a_2 - a_3 a_1|$

(Necessary + sufficient conditions for $n=3$)

Thm 2.12 If the solutions $\lambda_1, \ldots, \lambda_n$ of $p(\lambda)=0$, where

$p(\lambda) = \lambda^n + a_1 \lambda^{n-1} + \cdots + a_{n-1}\lambda + a_n$ satisfy $|\lambda_i| < 1$, then

(1) $p(1) = 1 + a_1 + a_2 + \cdots + a_n > 0$

(2) $(-1)^n p(-1) = 1 - a_1 + a_2 - \cdots + (-1)^n a_n > 0$

(3) $|a_n| < 1$.

(Necessary but not sufficient)

Def. 2.9 Let $\bar{X}$ be an equilibrium of $X(t+1) = F(X(t))$

and let $J(\bar{X})$ be the Jacobian matrix at $\bar{X}$. $\bar{X}$ is

hyperbolic if $|\lambda_i| \neq 1$ $\forall$ eigenvalues λ_i of $J(\bar{X})$.

Otherwise, it is nonhyperbolic.